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Linear Rayleigh-Taylor stability of viscous fluids with mass and heat transfer

By S-P. HO

Division of Applied Mathematics, Brown University, R.I. 02912

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The linear Rayleigh–Taylor stability of superposed viscous fluids with interfacial transfer of mass and heat is first considered for layers of finite thickness. A dispersion relation is obtained. It is then employed to derive stability and instability criteria for the case of two semi-infinite layers as well as the case where one of the layers is finite. From these criteria one arrives at a critical dispersion relation and a new critical wavenumber. This new critical wavenumber is distinct from the classical value owing to the presence of a parameter which depends, in a very simple manner, upon the kinematic viscosity of the fluids, the surface tension and the rate of interfacial transfer of mass and energy. Also it is found that the stabilizing effect of the surface tension is neither affected by the arrangement of the system nor the direction of the temperature gradient. However, the effects of the viscosity and the gravity will depend upon the relative positions of the superposed fluids and the direction of the temperature gradient at the interface.

1. Introduction

The problem of interfacial stability of superposed fluids has been investigated since the turn of the century by Rayleigh (1900), Harrison (1908), Taylor (1950), Lewis (1950) and Chandrasekhar (1955), who assumed that interfacial transfer of mass and energy could be neglected and used linear analysis. Interest in this problem subsided during the sixties but was revived in the early seventies. A very comprehensive analysis for inviscid fluids with interfacial transfer of mass and heat was given by Hsieh (1972, 1978); meanwhile an investigation of the marginal instability at an interface of a rapidly evaporating pure liquid under reduced pressure was carried out by Palmer (1976). The investigation of Palmer dealt essentially with the case where there was a very intense transfer of heat and mass across the vapour–liquid interface, and where the vapour density might be neglected. The present study will deal with the case of two superposed fluids of different densities where the density of the lighter fluid is not negligible and the interfacial transfer of mass and energy is not necessarily very intense.

As described in Hsieh's papers (1972, 1978), amongst other variations of the problem of interfacial stability, a situation where the effect of heat and mass transfer across the interface will be important arises from film-boiling as well as pool-boiling heat transfer. An example with some experimental results was provided by Dhir & Lienhard (see Hsieh 1972).

Another situation where the present study may find an application is in the area

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of secondary oil recovery, where one may consider the cyclic oil-drop process (instead of the bubble-escapement process) in which the unstable interfacial waves grow and collapse to release oil drops.

Furthermore the new critical dispersion relation,

$$k^2 + \frac{4\alpha\nu}{\sigma} k - k_c^2 = 0,$$

offers an explanation of why it evaded both the investigation of Chandrasekhar (who neglected α) and that of Hsieh (who neglected ν), where α, ν, σ and k_c are the rate of interfacial mass and heat transfer, kinematic viscosity, surface-tension coefficient and the classical critical wavenumber respectively. It is because α and ν are coupled together in the critical case, that neglecting one would mean the disappearance of the other. In the present study the method of approach to the problem may be outlined as follows.

The problem is first converted to a boundary-valued problem in two adjoining regions with a common boundary. The solution in each region is matched at the interface to conserve mass, momentum and energy. The requirement of a non-trivial solution in each fluid leads to a dispersion relation in the form of a transcendental equation which then enables the deduction of dispersion relations as quartic and cubic equations for the case of two semi-infinite layers and the case of one layer finite, respectively. A study of the nature of the roots of these polynomial equations leads to criteria of interfacial stability. That is, a small sinusoidal wave at the interface will grow or decay with time in accordance with certain necessary and sufficient conditions. Finally these criteria are employed to derive a critical dispersion relation and a new critical wavenumber which is different from the well-known classical value, viz. if $\rho^{(1)} > \rho^{(2)}$

$$k_c = \left\{\frac{(\rho^{(1)} - \rho^{(2)})\,g}{\sigma}\right\}^{\frac{1}{2}},$$

where g is the gravity constant, and $\rho^{(1)}$, $\rho^{(2)}$ and σ are the densities and the surface-tension coefficient respectively.

2. Formulation

We shall consider two superposed fluids of densities $\rho^{(1)}$ and $\rho^{(2)}$, and of viscosities $\mu^{(1)}$ and $\mu^{(2)}$ such that at equilibrium the interface lies in the plane $y = 0$. The temperatures at the walls are T_1 and T_2. Thus figure 1 summarizes the geometry of the problem.

Let the wavelike profile of the interface be represented by

$$S(x,y,t) \equiv y - \eta(x,t) = 0, \tag{1}$$

where $\eta(x,t) = \zeta e^{i\theta}$, $\theta = i(kx+nt)$, ζ is the amplitude, k is the wavenumber of the interfacial wave and n is a constant. If the perturbed velocity components and the pressure are

$$u = \tilde{u}(y)\, e^{i\theta}, \quad v = w(y)\, e^{i\theta}, \quad p = \tilde{p}(y)\, e^{i\theta},$$

then, substituting these into the Navier–Stokes equations for the conservation of momentum and linearizing the two differential equations, with the aid of the equation

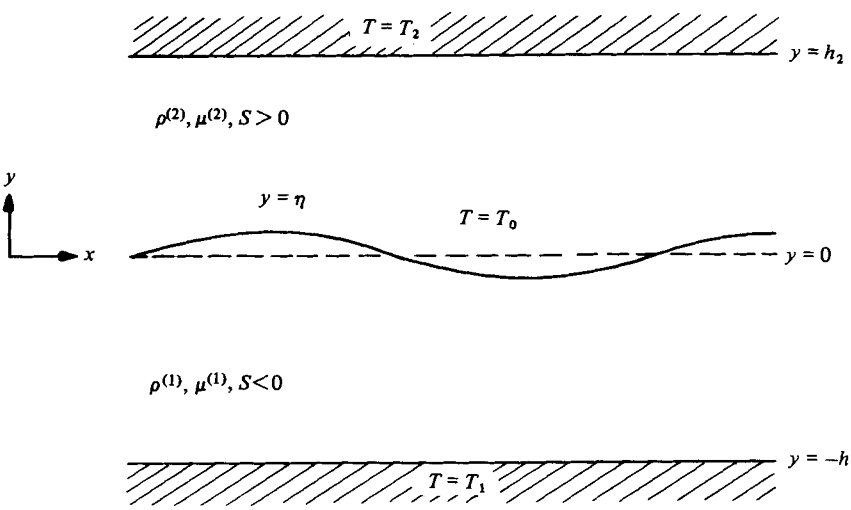

FIGURE 1. The problem considered.

of continuity (i.e. the conservation of mass), one obtains a differential equation for w, the amplitude of the vertical component of the perturbed velocity:

$$\left[1 - \frac{\mu}{in\rho}\left(\frac{d^2}{dy^2} - k^2\right)\right]\left(\frac{d^2}{dy^2} - k^2\right) w = 0. \tag{2}$$

Subject to the boundary conditions that $w = dw/dy = 0$ on $y = -h_1$ and $y = h_2$, it can be shown that the solution of (2) is

$$w^{(1)} = A_1 (\cosh q_1 \phi - \cosh k\phi) + B_1(q_1 \sinh k\phi - k \sinh q_1 \phi), \qquad -h_1 \leqslant y < \eta; \tag{3}$$

$$w^{(2)} = A_2 (\cosh q_2 \psi - \cosh k\psi) + B_2(q_2 \sinh k\psi - k \sinh q_2 \psi), \qquad \eta < y \leqslant h_2. \tag{4}$$

Here $$\phi = y + h_1, \quad \psi = y - h_2; \quad q_j = \left[k^2 + \frac{in}{\nu^{(j)}}\right]^{\frac{1}{2}}, \quad \nu^{(j)} = \frac{\mu^{(j)}}{\rho^{(j)}}, \quad j = 1, 2;$$

and A_1, B_1, A_2 and B_2 are constants.

Interfacial conditions. Using the simplified formulation of Hsieh (1972, 1978), the conservations of mass, momentum and energy across the interface lead to

$$\rho^{(1)}\left(\frac{\partial S}{\partial t} + u^{(1)}\frac{\partial S}{\partial x} + v^{(1)}\frac{\partial S}{\partial y}\right) = \{(2)\}, \tag{5}$$

$$\rho^{(1)}u^{(1)}\left(\frac{\partial S}{\partial t} + u^{(1)}\frac{\partial S}{\partial x} + v^{(1)}\frac{\partial S}{\partial y}\right) - \tau^{(1)}_{xx}\frac{\partial S}{\partial x} - \tau^{(1)}_{yx}\frac{\partial S}{\partial y} = \{(2)\} - \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\partial S}{\partial x}, \tag{6}$$

$$\rho^{(1)}v^{(1)}\left(\frac{\partial S}{\partial t} + u^{(1)}\frac{\partial S}{\partial x} + v^{(1)}\frac{\partial S}{\partial y}\right) - \tau^{(1)}_{xy}\frac{\partial S}{\partial x} - \tau^{(1)}_{yy}\frac{\partial S}{\partial y} = \{(2)\} - \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\partial S}{\partial y}, \tag{7}$$

$$L\rho^{(1)}\left(\frac{\partial}{\partial t} + u^{(1)}\frac{\partial S}{\partial x} + v^{(1)}\frac{\partial S}{\partial y}\right) = F(\eta), \tag{8}$$

where $\{(2)\}$ is an expression exactly the same as the one on the left but for the superscript 1 being replaced by 2; thus $u^{(1)}$ and $v^{(1)}$ are respectively the horizontal and vertical components of velocity in region 1, and $\tau^{(1)}_{xx}$, $\tau^{(1)}_{yx}$, $\tau^{(1)}_{xy}$, $\tau^{(1)}_{yy}$ the components of the Cartesian stress tensor in region 1, referred to figure 1; $u^{(2)}$, $v^{(2)}$, $\tau^{(2)}_{xx}$, $\tau^{(2)}_{yx}$, $\tau^{(2)}_{xy}$ and $\tau^{(2)}_{yy}$ are similarly defined quantities in region 2; $1/R_1+1/R_2$ is the curvature of the interface, L is the latent heat of transformation from the fluid of density $\rho^{(1)}$ to the fluid of density $\rho^{(2)}$; and $F(\eta)$ is a function of the instantaneous profile of the interface, and is determined from the heat-transfer relation at equilibrium (cf. Hsieh 1978). Physically the left-hand side of (8) represents the latent heat released during the phase transformation, while $F(\eta)$ on the right-hand side of (8) represents the net heat flux across the interface, so that energy will be conserved. By neglecting nonlinear terms, one obtains, from (6),

$$\mu^{(1)}\left(\frac{d^2w^{(1)}}{dy^2}+k^2w^{(1)}\right) = \{(2)\},$$

which is the same as the condition of continuity of tangential stresses across the interface (cf. Chandrasekhar 1961). From (5)–(8) and the no-slip condition across the interface, neglecting nonlinear terms and suppressing the factor $e^{i\theta}$ give the following interfacial conditions:

$$\frac{dw^{(1)}}{dy}=\frac{dw^{(2)}}{dy}, \tag{9}$$

$$\mu^{(1)}\left[\frac{d^2w^{(1)}}{dy^2}+k^2w^{(1)}\right] = \mu^{(2)}\left[\frac{d^2w^{(2)}}{dy^2}+k^2w^{(2)}\right], \tag{10}$$

$$\rho^{(1)}_{*}\,w^{(1)} = \rho^{(1)}_{*}\,w^{(2)}, \tag{11}$$

$$\sigma^*\rho^{(1)}_{*}\,w^{(1)}-\mu^{(1)}_{*}\frac{dw^{(1)}}{dy}+\frac{\mu^{(1)}}{k^2}\frac{d^3w^{(1)}}{dy^3} = -\mu^{(2)}_{*}\frac{dw^{(2)}}{dy}+\frac{\mu^{(2)}}{k^2}\frac{d^3w^{(2)}}{dy^3}; \tag{12}$$

in the above $\alpha = F'(0)/L$, $F'(0)$ is the derivative of $F(\eta)$ at $\eta=0$; $\sigma^* = (\rho^{(2)}-\rho^{(1)})g-k^2\sigma$; $\rho^{(1)}_{*} = \rho^{(1)}/(\alpha+in\rho^{(1)})$, $\mu^{(1)}_{*} = 3\mu^{(1)}+(in/k^2)\,\rho^{(1)}$ in $-h_1 \leqslant y < \eta$, $i^2=-1$; $\rho^{(2)}_{*}$ and $\mu^{(2)}_{*}$ are similarly defined in $\eta < y \leqslant h_2$. For (3) and (4) to be the solution of the problem under consideration, they must satisfy the interfacial conditions (9)–(12).

3. Dispersion relation

Substitution of solution (3) and (4) in the interfacial conditions (9)–(12) leads to the following system of algebraic equations for A_1, B_1, A_2 and B_2:

$$a^{(1)}A_1+kq_1b^{(1)}B_1+a^{(2)}A_2-kq_2b^{(2)}B_2 = 0,$$

$$c^{(1)}A_1+d^{(1)}B_1-c^{(2)}A_2+d^{(2)}B_2 = 0,$$

$$-\rho^{(1)}_{*}\,b^{(1)}A_1+\rho^{(1)}_{*}\,e^{(1)}B_1+\rho^{(2)}_{*}\,b^{(2)}A_2+\rho^{(2)}_{*}\,e^{(2)}B_2 = 0,$$

$$\begin{aligned}(-\sigma^*\rho^{(1)}_{*}\,b^{(1)}-\mu^{(1)}_{*}\,a^{(1)}+f^{(1)})\,A_1+(\sigma^*\rho^{(1)}_{*}\,e^{(1)}-\mu^{(1)}_{*}\,b^{(1)}+l^{(1)})\,B_1\\ +(-\mu^{(2)}_{*}\,a^{(2)}+f^{(2)})\,A_2+(\mu^{(2)}_{*}\,kq_2b^{(2)}-l^{(2)})\,B_2 = 0;\end{aligned}$$

where

$$a^{(1)} = q_1\sinh q_1h_1-k\sinh kh_1,$$

$$b^{(1)} = \cosh kh_1-\cosh q_1h_1,$$

$$c^{(1)} = \mu^{(1)}[(q_1^2+k^2)\cosh q_1h_1-2k^2\cosh kh_1],$$

$$d^{(1)} = \mu^{(1)}[2k^2q_1\sinh kh_1-k(q_1^2+k^2)\sinh q_1h_1],$$

$$e^{(1)} = q_1\sinh kh_1-k\sinh q_1h_1,$$

$$f^{(1)} = \frac{\mu^{(1)}q_1}{k^2}(q_1^3 \sinh q_1 h_1 - k^3 \sinh kh_1),$$

$$l^{(1)} = \frac{\mu^{(1)}q_1}{k}(k^2 \cosh kh_1 - q_1^2 \cosh q_1 h_1);$$

$a^{(2)}$, $b^{(2)}$, $c^{(2)}$, $d^{(2)}$, $e^{(2)}$, $f^{(2)}$ and $l^{(2)}$ are similarly defined. In order that A_1, B_1, A_2 and B_2 are not all zero, the determinant of the coefficient matrix of the above system of algebraic equations must vanish; i.e.

$$\Phi(n, k; \alpha, h_1, h_2; \sigma, \nu^{(1)}, \nu^{(2)}, \rho^{(1)}, \rho^{(2)}) = 0. \tag{13}$$

Equation (13) is given explicitly in appendix A and is the dispersion relation for two finite layers of viscous fluids, superposed above one another as shown in figure 1.

If $h_1 \to \infty$ and $h_2 \to \infty$, (13) becomes

$$\Phi(n, k; \alpha, \infty, \infty; \sigma, \nu^{(1)}, \nu^{(2)}, \rho^{(1)}, \rho^{(2)}) = 0,$$

which is identical with the dispersion relation, obtained by Chandrasekhar (1961), when $\alpha \to 0$, i.e. when the interfacial transfer of heat and mass is neglected. The same result could also be obtained by considering directly two semi-infinite layers of viscous fluids superposed above one another. Again one will recover the result of Chandrasekhar upon neglecting interfacial heat and mass transfer.

4. Interfacial stability when $\alpha > 0$ (α defined in equation (12))

Case (1): Two semi-infinite layers

Letting $h_1 \to \infty$, $h_2 \to \infty$, $\nu^{(1)} = \nu^{(2)} = \nu$ and $z = in/\nu$, then $q_1 = q_2 = q = (k^2+z)^{\frac{1}{2}}$ and dispersion relation (13) becomes a quartic equation in z, viz.

$$b_0 z^4 + b_1 z^3 + b_2 z^2 + b_3 z + b_4 = 0, \tag{14}$$

where

$$b_0 = (\rho^{(1)} + \rho^{(2)})^2,$$

$$b_1 = 4(\rho^{(1)} + \rho^{(2)})\left\{\frac{\alpha}{\nu} + 2k^2[a + \alpha\rho^{(2)}(1 - 2\alpha_1\alpha_2)]\right\},$$

$$b_2 = 4\left(\frac{\alpha}{\nu} + 2k^2 a\right)\left[\frac{\alpha}{\nu} + k^2(2a - \rho^{(1)} - \rho^{(2)})\right] + 2k(\rho^{(1)} + \rho^{(2)})\left(8k^3 a - \frac{\sigma^*}{\nu^2}\right) - 16k^4 a^2,$$

$$b_3 = 4k^2\left(\frac{\alpha}{\nu} + 2k^2 a\right)^2 + 2k\left[\frac{2}{\nu}\alpha + k^2(3a + \rho^{(1)} + \rho^{(2)})\right]\left(8k^3 a - \frac{\sigma^*}{\nu^2}\right) - 64k^6 a^2,$$

$$b_4 = k^2\left(8k^3 a - \frac{\sigma^*}{\nu^2}\right)\left(\frac{4k\alpha}{\nu} - \frac{\sigma^*}{\nu^2}\right),$$

$$a = \frac{(\rho^{(2)} - \rho^{(1)})^2}{\rho^{(1)} + \rho^{(2)}},$$

$$\alpha_1 = \frac{\rho^{(1)}}{\rho^{(1)} + \rho^{(2)}},$$

$$\alpha_2 = \frac{\rho^{(2)}}{\rho^{(1)} + \rho^{(2)}}.$$

Since (14) is a real quartic equation in z, $b_4 < 0$ implies that there are at least one positive and one negative real root (Burnside & Panton 1892). But each perturbed quantity of the flow has been multiplied by a factor $\exp(ikx + \nu zt)$, so a positive real root of (14) corresponds to an unstable mode of the interfacial disturbance, which will grow exponentially with time. If the coefficient $b_4 < 0$, then either

$$\frac{4ka}{\nu} < \frac{\sigma^*}{\nu^2} < 8k^3\alpha \qquad (15a)$$

or

$$8k^3a < \frac{\sigma^*}{\nu^2} < \frac{4k\alpha}{\nu}. \qquad (15b)$$

These are sufficient conditions for interfacial instability. We shall return to them later.

By examining the roots of the quartic (14), one obtains criteria for interfacial stability. It is well known that, if all the roots of a quartic equation are negative, all the coefficients of the equation are positive (Burnside & Panton 1892; Beuriger 1901). However, one can show (cf. appendix B) that the converse is not necessarily true and that the sufficient condition for all the roots (or their real parts) of (14) to be negative are

$$b_1b_2b_3 - b_0b_3^2 - b_1^2b_4 > 0, \quad b_4/b_0 > 0. \qquad (16)$$

Substituting the coefficients b_0, b_1, b_2, b_3 and b_4 from (14) into (16) leads to

$$\begin{aligned} &2k^2\lambda^5 + [k^4(2+70\epsilon-\epsilon^2) - 4\tilde{\sigma}^*]\lambda^4 + 2k[k^4(1+56\epsilon+327\epsilon^2-16\epsilon^3) \\ &\quad -(3+30\epsilon-\epsilon^2)\tilde{\sigma}^*]\lambda^3 + [8k^8(7\epsilon+152\epsilon^2+193\epsilon^3-24\epsilon^4) \\ &\quad + k^4(9-138\epsilon-260\epsilon^2+22\epsilon^3)\tilde{\sigma}^* + \tfrac{1}{4}(1-3\epsilon)\tilde{\sigma}^{*2}]\lambda^2 \\ &\quad + k^2[16k^8(\epsilon+44\epsilon^2+288\epsilon^3-28\epsilon^4-\epsilon^5) - 2k^4(1-6\epsilon+360\epsilon^2+162\epsilon^3-25\epsilon^4)\tilde{\sigma}^* \\ &\quad + (1-\epsilon-4\epsilon^2)\tilde{\sigma}^{*2}]\lambda + k^4[128k^8\epsilon^2(1+24\epsilon-3\epsilon^2) \\ &\quad - 8k^4\epsilon(1+32\epsilon+88\epsilon^2-2\epsilon^3-\epsilon^4)\tilde{\sigma}^* - (1-4\epsilon+6\epsilon^2+4\epsilon^3+\epsilon^4)\tilde{\sigma}^{*2}] > 0, \end{aligned} \qquad (17)$$

and

$$8k^3a > \frac{4k\alpha}{\nu} > \frac{\sigma^*}{\nu^2}, \qquad (18)$$

where

$$\epsilon = (\rho^{(2)}-\rho^{(1)})^2/(\rho^{(2)}+\rho^{(1)})^2, \quad \lambda = \frac{2\alpha}{\nu(\rho^{(1)}+\rho^{(2)})}, \quad \tilde{\sigma}^* = \frac{k\sigma^*}{\nu^2(\rho^{(1)}+\rho^{(2)})}.$$

Since each negative root (or a complex root with a negative real part) of (14) corresponds to a stable mode of the interfacial disturbance, (17) and (18) which were obtained from (16) are the sufficient conditions that all the modes of the interfacial disturbance will be stable. But, from (18), one obtains

$$8k^3\nu^2\frac{(\rho^{(2)}-\rho^{(1)})^2}{\rho^{(1)}+\rho^{(2)}} > 4k\alpha\nu > (\rho^{(2)}-\rho^{(1)})g - k^2\sigma. \qquad (18)^*$$

Thus provided α satisfies (18)* and is sufficiently large, according to the coefficients b_0, b_1, b_2, b_3 and b_4 of (14) since $b_1b_2b_3 = O(\alpha^5)$, $b_0b_3^2 = O(\alpha^4)$ and $b_1^2b_4 = O(\alpha^3)$, equation (17), i.e. the first inequality of (16), will be satisfied. Therefore, if α satisfies (18)* and is sufficiently large, (17) may be relaxed. Hence one obtains from (18)* the stability criterion for interfacial disturbances, when α satisfies

$$k^2\sigma + 4k\alpha\nu - (\rho^{(2)}-\rho^{(1)})g > 0. \qquad (19)$$

Returning to the instability criteria (15*a*) and (15*b*), one can rewrite them as

$$4k\alpha\nu < (\rho^{(2)}-\rho^{(1)})g-k^2\sigma < 8k^3\nu^2\frac{(\rho^{(2)}-\rho^{(1)})^2}{\rho^{(1)}+\rho^{(2)}} \tag{15a*}$$

and

$$8k^3\nu^2\frac{(\rho^{(2)}-\rho^{(1)})^2}{\rho^{(1)}+\rho^{(2)}} < (\rho^{(2)}-\rho^{(1)})g-k^2\sigma < 4k\alpha\nu \tag{15b*}$$

respectively. Both (15*a*)* and (15*b*)* are compatible with the stability criterion (18)*, but from (15*a*)* one obtains

$$k^2\sigma+4k\alpha\nu-(\rho^{(2)}-\rho^{(1)})g < 0, \tag{20}$$

i.e. when there is interfacial transfer of heat and mass such that α satisfies (20), interfacial waves will be unstable.

In the case of inviscid fluids of finite thickness, i.e. $\nu^{(1)}\to 0$ and $\nu^{(2)}\to 0$, from the dispersion relation (13)

$$\Phi(n,k;\alpha,h_1,h_2;\sigma,0,0,\rho^{(1)},\rho^{(2)}) = 0$$

one recovers the established result, given by (23) in Hsieh's (1978) paper.

Case (2): *Finite and semi-infinite layers*

This is the case which has the physical realization of a very thick layer of liquid superposed on a thin layer of vapour. Let $\nu^{(1)}=\nu^{(2)}=\nu$. Then as $h_1\to 0$, $h_2\to\infty$, and, if the fourth- and higher-order terms of h_1 may be neglected, the dispersion relation (13) becomes

$$c_0z^3+c_1z^2+c_2z+c_3 = 0, \tag{21}$$

where

$$z = \frac{i}{\nu}n,$$

$$c_0 = h_1^2(a+2bkh_1),$$

$$c_1 = \frac{\alpha h_1}{\nu\rho^{(2)}}(2\gamma a+dkh_1)-\frac{1}{\gamma^2}-\frac{2kh_1}{\gamma}+\frac{2k^2h_1^2}{\gamma}(2b-1)+\tfrac{4}{3}k^3h_1^3(4b-1),$$

$$c_2 = \left(\frac{\alpha h_1}{\nu\rho^{(2)}}\right)^2\left[\gamma^2a+kh_1\gamma\left(\gamma a-\frac{b}{3}\right)\right]+\frac{\alpha}{\nu\rho^{(2)}}\left[-\frac{2}{\gamma}-4kh_1+k^2h_1^2(7b-4)+ek^3h_1^3\right]+\frac{2}{\gamma}\eta^*,$$

$$c_3 = (F-\eta^*)(\eta^*-G),$$

$$F = \frac{\alpha}{\nu\rho^{(2)}}[1+2\gamma kh_1+2k^2h_1^2+\tfrac{2}{3}(\gamma+1)k^3h_1^3],$$

$$G = \frac{\alpha}{\nu\rho^{(2)}}[1-(\gamma-1)k^2h_1^2],$$

$$\eta^* = \frac{k^2h_1^3}{3}\frac{\sigma^*}{\nu^2\rho^{(2)}},\quad \gamma = \frac{\rho^{(2)}}{\rho^{(1)}},\quad a = 1-\frac{1}{\gamma^2},$$

$$b = 1-\frac{1}{\gamma},\quad d = 3\gamma-\frac{7}{3}-\left(\frac{3\gamma}{2}\right)^{-1},\quad e = 6\gamma-8-\frac{2}{3\gamma}.$$

To be consistent with the approximation of neglecting the fourth- and higher-order terms of h_1, it was assumed that $\eta^* = O(h_1^3)$, as $h_1\to 0$. Since $\gamma > 1$, $a > 0$ and

$b > 0$, therefore $c_0 > 0$. If $c_3 < 0$, the cubic equation (21) will have at least one positive real root (see, for example, Burnside & Panton 1892), and hence the interfacial wave will have at least one unstable mode. But if $c_3 < 0$ in (21), then η^* must satisfy either

$$G < F < \eta^* \quad \text{or} \quad \eta^* < G < F.$$

When $\alpha > 0$, F and G, defined in the above, clearly satisfy $G < F$. On the other hand, for σ^* defined in (12), if $\eta^* < G$,

$$\frac{k^2h_1^3}{3\nu}[(\rho^{(2)}-\rho^{(1)})g-k^2\sigma] < \alpha[1-(\gamma-1)k^2h_1^2],$$

which becomes $(\rho^{(2)}-\rho^{(1)})g-k^2\sigma < 0$, as $\alpha \to 0$ or when $\gamma = \rho^{(2)}/\rho^{(1)} = 1+1/(k^2h_1^2)$. But $\gamma = \rho^{(2)}/\rho^{(1)}$ is a dimensionless parameter which depends upon the densities of the fluids, while h_1, the thickness of the bottom fluid layer, depends only on the geometry of the problem. Thus $h_1 \to 0$ is equivalent to $\gamma \to \infty$ for any fixed k, according to $\gamma = 1+1/(k^2h_1^2)$. Therefore whenever $\gamma = 1+1/(k^2h_1^2)$, the right-hand side of the inequality vanishes, but the left-hand side can remain positive. This is a contradiction. Therefore $G < F < \eta^*$, i.e. $F < \eta^*$ or

$$\alpha[1+2\gamma kh_1+2k^2h_1^2+\tfrac{2}{3}(\gamma+1)k^3h_1^3] < \frac{k^2h_1^3}{3\nu}[(\rho^{(2)}-\rho^{(1)})g-k^2\sigma] \tag{22}$$

is the appropriate instability criterion, since the requirement that $G < F$ is superfluous. If now $\alpha \to 0$, one recovers from (22) the well-known classical instability criterion, namely

$$0 < (\rho^{(2)}-\rho^{(1)})g-k^2\sigma.$$

According to (22), the effects of the surface tension and the viscosity are both stabilizing but the gravity or, more precisely, $(\rho^{(2)}-\rho^{(1)})g$, the difference of the fluid densities, tends to destabilize the interfacial wave when $\rho^{(1)} < \rho^{(2)}$ as one would expect. Further, since, for sufficiently small h_1,

$$\frac{4k^3h_1^3}{3} < 1+2\gamma kh_1+2k^2h_1^2+\tfrac{2}{3}(\gamma+1)k^3h_1^3,$$

equation (22) implies that

$$\alpha\left(\frac{4k^3h_1^3}{3}\right) < \frac{k^2h_1^3}{3\nu}[(\rho^{(2)}-\rho^{(1)})g-k^2\sigma]$$

or

$$k^2\sigma+4k\alpha\nu-(\rho^{(2)}-\rho^{(1)})g < 0. \tag{23}$$

Hence the interfacial wave is unstable when the wavenumber k, and the interfacial heat and mass transfer α, satisfy (23).

It can easily be shown (see appendix C) that all the roots (or their real parts) of a real cubic are negative if and only if all the coefficients are positive (and $c_1c_2 > c_3$, if (21) has complex roots). Thus, according to (21), all the modes of an interfacial disturbance are stable if and only if c_0, c_1, c_2 and c_3 are positive (and $c_1c_2 > c_3$, if (21) has complex roots). When $\gamma > 1$, $a > 0$ and $b > 0$, therefore $c_0 > 0$. Also from (21), when $c_3 > 0$,

$$G < \eta^* < F,$$

i.e. for F, G, η^* and σ^*, defined in (21) and (12),

$$\alpha[1-(\gamma-1)k^2h_1^2] < \frac{k^2h_1^3}{3\nu}[(\rho^{(2)}-\rho^{(1)})g-k^2\sigma] < \alpha[1+2\gamma kh_1+2k^2h_1^2+\tfrac{2}{3}(\gamma+1)k^3h_1^3]. \tag{24}$$

It can be shown that, for $\gamma = O(h_1^{-2})$ and arbitrarily small h_1, if α satisfies (24), $c_1 > 0$, $c_2 > 0$ and $c_1 c_2 > c_3$ will also be satisfied. Therefore $c_3 > 0$ implies $c_1 > 0$, $c_2 > 0$ and $c_1 c_2 > c_3$; and (24) is the necessary and sufficient condition that interfacial disturbances will be stable.

The first inequality in (24) ensures that $0 < \eta^*$, so that the surface tension σ cannot be too large to inhibit completely the heat and mass transfer across the interface. Thus, for values of σ such that $0 < \eta^*$, the stability criterion is given by

$$\frac{k^2 h_1^3}{3\nu}[(\rho^{(2)} - \rho^{(1)})\, g - k^2\sigma] < \alpha[1 + 2\gamma k h_1 + 2k^2 h_1^2 + \tfrac{2}{3}(\gamma+1)\, k^3 h_1^3]. \tag{25}$$

Since from (25), for given $\rho^{(1)}$, $\rho^{(2)}$ and σ, there are values of α, k and h_1 such that the following estimate holds,

$$\frac{k^2 h_1^3}{3\nu}[(\rho^{(2)} - \rho^{(1)})\, g - k^2\sigma] < \alpha\left(\frac{4k^3 h_1^3}{3}\right) < \alpha[1 + 2\gamma k h_1 + 2k^2 h_1^2 + \tfrac{2}{3}(\gamma+1)\, k^3 h_1^3],$$

an interfacial wave is stable if the wavenumber k, and the interfacial heat and mass transfer α, satisfy

$$\frac{k^2 h_1^3}{3\nu}[(\rho^{(2)} - \rho^{(1)})\, g - k^2\sigma] < \alpha\left(\frac{4k^3 h_1^3}{3}\right),$$

i.e. if

$$k^2\sigma + 4k\alpha\nu - (\rho^{(2)} - \rho^{(1)})\, g > 0. \tag{26}$$

Consequently combining the results of (19), (20), (23) and (26) leads finally to the critical dispersion relation,

$$k^2\sigma + 4k\alpha\nu - (\rho^{(2)} - \rho^{(1)})\, g = 0, \tag{27}$$

and the corresponding critical wavenumber,

$$k_* = -\frac{2\alpha\nu}{\sigma} + \left(\frac{4\alpha^2\nu^2}{\sigma^2} + k_c^2\right)^{\frac{1}{2}},$$

where

$$k_c = \left\{\frac{(\rho^{(2)} - \rho^{(1)})}{\sigma}\, g\right\}^{\frac{1}{2}}$$

is the classical critical wavenumber. Thus when there is an interfacial transfer of mass and energy coupled by viscosities of the fluids, the interfacial disturbance is unstable if $k < k_*$ and it is stable if $k > k_*$.

If the next-higher-order terms, i.e. $O(h_1^4)$, are retained when approximation of the dispersion relation (13) is taken, one will obtain a quartic or biquadratic equation as the dispersion relation, but will arrive at the same result as before.

5. Interfacial stability when $\alpha < 0$ (α defined in equation (12))

This corresponds to the case where the direction of the temperature gradient is from the denser to the less dense fluid. In order to relate Palmer's (1976) investigation with the instability criterion obtained earlier, it is desirable to explain briefly the terminology of 'vapour recoils' and 'induced convections', referred to in Palmer's (1976) paper. Whenever there is evaporation, the energy difference between molecules in the liquid phase and those in the vapour phase must balance the latent heat of transformation released or absorbed at the interface. As the vapour molecules escape from the interface, each molecule leaves behind an equal but opposite amount of momentum. This is known as the 'vapour-recoil'. Referring to figure 1, let a crest be defined as a point on the interface, which has the maximum displacement above the x axis when $\rho^{(1)} > \rho^{(2)}$, and the crest will be below the x axis when $\rho^{(1)} < \rho^{(2)}$. A trough is the mirror image of a crest with respect to the x axis. As a result of the wavelike profile, the vapour pressure is not uniform along the interface, since at a crest it is smaller while at a trough it is larger than its equilibrium value at $y = 0$. Such a difference of vapour pressure along the interface gives rise to a non-uniform distribution of the rate of evaporation, and hence a non-uniform distribution of vapour-recoils. This, in turn, generates induced convections or vortices on either side of and immediately adjacent to the interface. These induced convections will grow or decay with time, depending upon the rate of evaporation and how effectively momenta can be transmitted from the vapour-recoils to the denser fluid. However, the surface tension will remain uniform for linear analysis by arguments similar to those of uniform tension in linear vibrations of an elastic membrane or string.

Referring to figure 1, the system considered by Palmer (1976) corresponds to: $h_1 \to \infty, h_2 \to \infty; \alpha < 0$, i.e. a cooler vapour superposed on a hotter liquid; and $\rho^{(2)} \ll \rho^{(1)}$, i.e. the vapour being under reduced pressure. Let $\alpha = -\tau < 0$. Since $\rho^{(1)} > \rho^{(2)}$, therefore $\sigma^* = -(\rho^{(1)}-\rho^{(2)})g - k^2\sigma$ and from $(15a)^*$ the interfacial wave will be unstable if

$$4k\tau\nu > k^2\sigma + (\rho^{(1)}-\rho^{(2)})g > -8k^3\nu^2 \frac{(\rho^{(1)}-\rho^{(2)})^2}{\rho^{(1)}+\rho^{(2)}}. \tag{28}$$

Since $\tau > 0$ and $\rho^{(1)} > \rho^{(2)}$, the second inequality of (28) involving a negative term is superfluous, and the instability criterion is actually given by the first inequality of (28); hence the instability criterion becomes

$$4k\tau\nu > k^2\sigma + (\rho^{(1)}-\rho^{(2)})g.$$

From $\tau = |\alpha|$, τ is proportional to the rate of evaporation and hence to the vapour-recoil. Thus interfacial waves tend to be more unstable for larger τ. The destabilizing effect of vapour-recoils will be enhanced by the kinematic viscosity ν, for the latter will help the vortices to grow in size. This growth of vortices together with the temperature gradient in the fluids will enable transports of more hotter fluid elements to the interface from one side but more cooler fluid elements to the interface from the other. Thus there is an escalation of the temperature gradient across the interface. Consequently the interfacial disturbance tends to be more unstable. Rewriting the above inequality as

$$4\tau > \frac{1}{\nu}\left[k\sigma + \frac{(\rho^{(1)}-\rho^{(2)})g}{k}\right] \tag{28*}$$

shows that the difference of the fluid densities, $(\rho^{(1)}-\rho^{(2)})g$, tends to stabilize the interface. Apart from the stabilizing effect of gravity, stabilization can also arise from the effect of hotter liquid protruding into the cooler vapour region. There are two competing effects taking place simultaneously at the crest, namely evaporation and re-condensation. For the larger the value of $(\rho^{(1)}-\rho^{(2)})g$ the amount of latent heat released by condensation at the crests will be larger, which will then be used in heating up the vapour in the vicinity of the crests and thus enhancing the evaporation. On the other hand, as the cooler vapour tries to penetrate into the hotter liquid region, the larger the value of $(\rho^{(1)}-\rho^{(2)})g$ coupled by higher vapour pressure at the trough, the more rapid will be the condensation. From (28)* it is observed that the surface tension will remain as a stabilizing force whether $\alpha > 0$ or $\alpha < 0$. Furthermore the effects of the surface tension and the difference of fluid densities are wavenumber dependent; for short wavelengths (or large k) stabilization is mainly due to the surface tension σ, but for long wavelengths (or small k) the difference of fluid densities $(\rho^{(1)}-\rho^{(2)})g$ will have a more important stabilizing effect. These stabilizing and destabilizing mechanisms agree in general with those suggested by Palmer (1976).

Finally when $\alpha < 0$, the interfacial disturbance will always be unstable as pointed out by Hsieh (1978). We shall proceed to show this as follows.

Let $\alpha = -\tau < 0$. Then it can be shown that the first inequality of (16) will not hold, i.e. (17) will not be satisfied, when $\alpha < 0$. Substituting $\alpha = -\tau$ in the coefficients of (14) gives

$$
\begin{aligned}
b_0 &= (\rho^{(1)}+\rho^{(2)})^2,\\
b_1 &= 4(\rho^{(1)}+\rho^{(2)})\left\{-\frac{\tau}{\nu}+2k^2[a-\tau\rho^{(2)}(1-2\alpha_1\alpha_2)]\right\},\\
b_2 &= 4\left(-\frac{\tau}{\nu}+2k^2a\right)\left[-\frac{\tau}{\nu}+k^2(2a-\rho^{(1)}-\rho^{(2)})\right]+2k(\rho^{(1)}+\rho^{(2)})\left(8k^3a+\frac{\tilde{\sigma}}{\nu^2}\right)-16k^4a^2,\\
b_3 &= 4k^2\left(-\frac{\tau}{\nu}+2k^2a\right)^2+2k\left[-\frac{2}{\nu}\tau+k^2(3a+\rho^{(1)}+\rho^{(2)})\right]\left(8k^3a+\frac{\tilde{\sigma}}{\nu^2}\right)-64k^6a^2,\\
b_4 &= k^2\left(8k^3a+\frac{\tilde{\sigma}}{\nu^2}\right)\left(-\frac{4k\tau}{\nu}+\frac{\tilde{\sigma}}{\nu^2}\right),\\
\alpha_1 &= \rho^{(1)}/(\rho^{(1)}+\rho^{(2)}),\\
\alpha_2 &= \rho^{(2)}/(\rho^{(1)}+\rho^{(2)}),
\end{aligned}
$$

where only σ^* has to be replaced by $\sigma^* = -[(\rho^{(1)}-\rho^{(2)})g+k^2\sigma] = -\tilde{\sigma} < 0$; otherwise the coefficients b_0, b_1, b_2, b_3 and b_4 remain unchanged. That $b_4 > 0$ now implies

$$\frac{\tilde{\sigma}}{\nu^2} > \frac{4k\tau}{\nu} \quad \text{or} \quad \tilde{\sigma} > 4k\tau\nu.$$

For values of τ and ν satisfying the above inequality, since now $b_1b_2b_3$, $b_0b_3^2$ and $b_1^2b_4$ of (16) will be of the same order, $O(\nu^{-6})$, (17) cannot be relaxed as it could previously. But, as defined in (17) and (18),

$$\lambda = -\frac{2\tau}{\nu(\rho^{(1)}+\rho^{(2)})} \quad \tilde{\sigma}^* = -\frac{k\tilde{\sigma}}{\nu^2(\rho^{(1)}+\rho^{(2)})}, \quad 0 < \epsilon < 1,$$

therefore λ^2 and higher-order terms may be neglected as $\tau \to 0$ for a fixed ν and (17) reduces to

$$
\begin{aligned}
&\left[16k^8(\epsilon+44\epsilon^2+288\epsilon^3-28\epsilon^4-\epsilon^5)+2k^4(1-6\epsilon+360\epsilon^2+162\epsilon^3-25\epsilon^4)\frac{k\tilde{\sigma}}{\nu^2(\rho^{(1)}+\rho^{(2)})}\right.\\
&\qquad\left.+(1-\epsilon-4\epsilon^2)\frac{k^2\tilde{\sigma}^2}{\nu^4(\rho^{(1)}+\rho^{(2)})^2}\right]\left[-\frac{2\tau}{\nu(\rho^{(1)}+\rho^{(2)})}\right]\\
&\qquad+k^2\left[128k^8\epsilon^2(1+24\epsilon-3\epsilon^2)+8k^4\epsilon(1+32\epsilon+88\epsilon^2-2\epsilon^3-\epsilon^4)\frac{k\tilde{\sigma}}{\nu^2(\rho^{(1)}+\rho^{(2)})}\right.\\
&\qquad\qquad\left.-(1-4\epsilon+6\epsilon^2+4\epsilon^3+\epsilon^4)\frac{k^2\tilde{\sigma}^2}{\nu^4(\rho^{(1)}+\rho^{(2)})^2}\right] > 0.
\end{aligned}
$$

Letting $\nu \to 0$, the above condition becomes

$$-\frac{2k^2\tilde{\sigma}^2\tau}{\nu(\rho^{(1)}+\rho^{(2)})^3}(1-\epsilon-4\epsilon^2)-\frac{k^4\tilde{\sigma}^2}{(\rho^{(1)}+\rho^{(2)})^2}(1-4\epsilon+6\epsilon^2+4\epsilon^3+\epsilon^4) > 0$$

or

$$\frac{2\tau}{\nu(\rho^{(1)}+\rho^{(2)})}(1-\epsilon-4\epsilon^2)+k^2(1-4\epsilon+6\epsilon^2+4\epsilon^3+\epsilon^4) < 0$$

which is a contradiction, since the left-hand side of the above inequality should be positive for $0 < \epsilon < 1$. Thus the sufficient condition for interfacial stability (17) does not hold in this case. Therefore not all the roots of (14) are negative and the interfacial disturbance will be unstable.

So far we have been mainly concerned with cases of $\alpha > 0$, but what has just been discussed was essentially a case of two semi-infinite layers of fluids of different densities, superposed above one another, such that the upper layer was maintained at lower temperatures and the lower layer at higher temperatures. This is an example of the case $\alpha < 0$. Before concluding the present study, it may suffice to consider the case of a finite layer of cooler vapour on top of a semi-infinite layer of hotter liquid. Without going through all the algebraic manipulations again, one can make use of the earlier results as follows. Referring to figure 1, it can be shown that if $\rho^{(1)} > \rho^{(2)}$ and as $h_1 \to \infty$ and $h_2 \to 0$ in the derivation of (21) one will arrive at the same dispersion relation as (21) except that γ and η^* now become

$$\gamma = \frac{\rho^{(1)}}{\rho^{(2)}} > 1 \quad \text{and} \quad \eta^* = -\frac{k^2h_2^3}{3\nu^2\rho^{(1)}}[(\rho^{(1)}-\rho^{(2)})g+k^2\sigma].$$

Let $\alpha = -\tau < 0$. Then by means of (22), i.e. $F < \eta^*$, one obtains the following instability criterion:

$$\tau[1+2\gamma kh_2+2k^2h_2^2+\tfrac{2}{3}(\gamma+1)k^3h_2^3] > \frac{k^2h_2^3}{3\nu}[k^2\sigma+(\rho^{(1)}-\rho^{(2)})g]. \tag{29}$$

Again it can be shown that a stability criterion does not exist in this case. Let $\alpha = -\tau < 0$ be substituted in the coefficients of (21) and $\rho^{(1)}$ and $\rho^{(2)}$, h_1 and h_2 be interchanged. Then

$$c_0 = h_2^2(a+2bkh_2),$$

$$c_1 = -\frac{\tau h_2^2}{\nu\rho^{(1)}}(2\gamma a+dkh_2)-\frac{1}{\gamma^2}-\frac{2kh_2}{\gamma}+\frac{2k^2h_2^2}{\gamma}(2b-1)+\tfrac{4}{3}k^3h_2^3(4b-1),$$

$$c_2 = \left(\frac{\tau h_2}{\nu\rho^{(1)}}\right)^2\left[\gamma^2a+kh_2\gamma\left(\gamma a-\frac{b}{3}\right)\right]+\frac{\tau}{\nu\rho^{(1)}}\left[\frac{2}{\gamma}+4kh_2-k^2h_2^2(7b-4)-ek^3h_2^3\right]-\frac{2}{\gamma}\tilde{\eta},$$

$$c_3 = (F^*-\tilde{\eta})(\tilde{\eta}-G^*),$$

where $F^* = -F$, $G^* = -G$ and $\tilde{\eta} = -\eta^*$. Thus $c_3 > 0$ implies $G^* < \tilde{\eta} < F^*$. For τ, σ, ν and h_2 within these intervals, c_0, c_2 and c_3 are positive, but c_1 will be negative. Therefore $c_1c_2 \not> c_3$ and the necessary and sufficient condition, viz. that all the roots (or their real parts) of a real cubic are negative, will not be satisfied (see appendix C). Hence not all the modes of an interfacial wave are stable.

One may now conclude that, when $\alpha < 0$, an interfacial disturbance is always unstable. According to (28)* and (29) the kinematic viscosity ν tends to increase the growth rate of instability while the difference of fluid densities, $(\rho^{(1)}-\rho^{(2)})g$, tends to reduce the instability no matter whether the upper layer is finite or infinite.

If $\nu \neq 0$ (i.e. for viscous fluids), the viscosity will enhance interfacial instability because an increase in the viscosity ν implies a decrease in τ, according to (28)* and (29). In other words, the interfacial transfer of mass and heat will be inhibited by viscosity to such an extent that a steady build-up of heat at the interface occurs, because of the existence of temperature gradient in the fluids. Consequently an interfacial instability may be triggered off by any small disturbance. Thus, when the temperature gradient is directed from the denser to the less dense fluid, there will be no critical dispersion relation and hence no critical wavenumber. The stabilizing effect of the viscosity will become destabilizing as a result of reversing the temperature gradient, i.e. a change from $\alpha > 0$ (corresponding to $\rho^{(1)} < \rho^{(2)}$) to $\alpha < 0$ (corresponding to $\rho^{(1)} > \rho^{(2)}$), for in the latter case the vapour-recoil and the gravity, both pointing downwards, reinforce each other. But in the former case both the gravity and the viscosity tend to oppose the vapour-recoil and thus suppress the growth of vortices which, besides escalating the temperature gradient across the interface, will also tend to lower the surface tension.

6. Conclusion

The above study shows that the effect of the surface tension always stabilizes an interfacial disturbance no matter whether $\alpha > 0$ or $\alpha < 0$, where the sign of α, defined in (12), will indicate the direction of the temperature gradient across the interface. However, the viscosity tends to stabilize an interfacial disturbance if $\alpha > 0$ but destabilizes it if $\alpha < 0$; and, referring to figure 1, the gravity tends to destabilize the system when $\rho^{(1)} < \rho^{(2)}$ but stabilize it when $\rho^{(1)} > \rho^{(2)}$ with the stabilizing effect being enhanced by the interfacial transfer of mass and heat. For $\alpha \geqslant 0$, the kinematic viscosity ν of the fluids will affect the critical wavenumber. For the Rayleigh–Taylor stability problems we are considering, the critical wavenumber is given by

$$k_* = \left[\frac{4\alpha^2\nu^2}{\sigma^2} + k_c^2\right]^{\frac{1}{2}} - \frac{2\alpha\nu}{\sigma},$$

where
$$k_c = \left[\frac{|\rho^{(1)}-\rho^{(2)}|}{\sigma} g\right]^{\frac{1}{2}}, \quad \text{and} \quad \alpha > 0.$$

If $\nu \neq 0$, since $k_* - k_c < 0$ or $k_* < k_c$, the effect of viscosity is to lower the value of the critical wavenumber and hence enlarges the spectrum of wavenumbers over which interfacial disturbances will be stable.

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Appendix A

Expansion of the determinant of the coefficient matrix referred to on page 115 in the text leads to (13), which is a transcendental equation, viz.

$$\rho_*^{(1)} a^{(1)}(in\sigma^* \rho_*^{(2)} b^{(2)} - n^2 c^{(2)}) + \rho_*^{(2)} a^{(2)} (in\sigma^* \rho_*^{(1)} b^{(1)} - n^2 c^{(1)})$$
$$+ \rho_*^{(2)} d^{(1)}(f^{(2)} - e^{(2)}) + \rho_*^{(1)} d^{(2)}(e^{(1)} - f^{(1)}) + \rho_*^{(1)} b^{(1)} l^{(2)} + \rho_*^{(2)} b^{(2)} l^{(1)} = 0,$$

where $$a^{(1)} = \frac{k\rho^{(1)}}{2}[(q_1 - k)\sinh\theta_+ - (q_1+k)\sinh\theta_-],$$

$$b^{(1)} = -\tfrac{1}{2}[(q_1-k)^2\cosh\theta_+ - (q_1+k)^2\cosh\theta_- + 4kq_1],$$

$$c^{(1)} = \frac{\rho^{(1)}q_1}{2k}[(q_1-k)\sinh\theta_+ - (q_1+k)\sinh\theta_-],$$

$$d^{(1)} = \frac{k\mu^{(1)}}{2}[(q_1-k)^3\cosh\theta_+ + (q_1+k)^3\cosh\theta_- - 2q_1(q_1^2+3k^2)],$$

$$e^{(1)} = \frac{q_1}{2}[2k^2(\mu_*^{(1)} - \mu_*^{(2)}) - \mu^{(1)}(q_1^2+k^2)](\cosh\theta_+ + \cosh\theta_- - 2),$$

$$f^{(1)} = \frac{1}{2k}[k^2(\mu_*^{(1)} - \mu_*^{(2)})(q_1^2+k^2) - \mu^{(1)}(q_1^4+k^4)](\cosh\theta_+ - \cosh\theta_-),$$

$$l^{(1)} = \frac{k\mu^{(1)2}}{2}\Big[(3kq_1^3 + k^3q_1 - 2q_1^4 - k^2q_1^2 - k^4)\cosh\theta_+$$
$$+ (3kq_1^3 + k^3q_1 + 2q_1^4 + k^2q_1^2 + k^4)\cosh\theta_- - \frac{2q_1}{k}(q_1^4 + k^2q_1^2 + 2k^4)\Big],$$

$$\theta_+ = (q_1+k)h_1, \quad \theta_- = (q_1-k)h_1;$$

and $a^{(2)}$, $b^{(2)}$, $c^{(2)}$, $d^{(2)}$, $e^{(2)}$, $f^{(2)}$ and $l^{(2)}$ are similarly defined with $\rho^{(1)}$, $\mu^{(1)}$, q_1 and h_1 being replaced by $\rho^{(2)}$, $\mu^{(2)}$, q_2 and h_2 respectively.

Appendix B. Sufficient condition that all the roots (or their real parts) of a real quartic are negative

It has been well documented (Burnside & Panton 1892) that, if all the roots of a quartic are negative, all its coefficients will be positive. However, the converse is not true (Beuriger 1901), for example, although all the coefficients of the quartic

$$x^4 + x^3 + x^2 + x + 1$$

are positive, the real parts of two of its roots are positive and those of the other two negative. So it is desirable to find a sufficient condition for all the roots (or their real parts) to be negative. In general a real quartic,

$$b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4, \tag{A 1}$$

can be resolved into two quadratic factors (Bernside & Panton 1892),

$$\left[b_0x^2 + 2\left(\frac{b_1}{4} - M\right)x + \frac{b_2}{6} + 2b_0\theta - N\right]\left[b_0x^2 + 2\left(\frac{b_1}{4} + M\right)x + \frac{b_2}{6} + 2b_0\theta + N\right], \tag{A 2}$$

where
$$M^2 = \left(\frac{b_1}{4}\right)^2 - \frac{b_0 b_2}{6} + b_0^2\theta, \quad N^2 = \left(\frac{b_2}{6} + 2b_0\theta\right)^2 - b_0 b_4,$$
$$MN = \frac{b_1 b_2}{24} - \frac{b_0 b_3}{4} + \frac{b_0 b_1}{2}\theta,$$

and θ is a solution of the reducing cubic equation

$$4b_0^3\theta^3 - \left(b_0 b_4 - \frac{b_1 b_3}{4} + \frac{b_2^2}{12}\right) b_0\theta + \frac{b_0 b_2 b_4}{6} + \frac{b_1 b_2 b_3}{48} - \frac{b_0 b_3^2}{16} - \frac{b_1^2 b_4}{16} - \frac{b_2^3}{216} = 0. \quad \text{(A 3)}$$

Therefore the four roots of (A 1) are

$$x = \frac{1}{b_0}\left\{-\left(\frac{b_1}{4} - M\right) \pm \left[\left(\frac{b_1}{4} - M\right)^2 - b_0\left(\frac{b_2}{6} + 2b_0\theta - N\right)\right]^{\frac{1}{2}}\right\}$$

and a similar expression with $-M$ and $-N$ replaced by M and N. By writing $\frac{1}{4}b_1 \pm M$, $b_0(\frac{1}{6}b_2 + 2b_0\theta \pm N)$ and the square root into real and imaginary parts and comparing the real part of $\frac{1}{4}b_1 \pm M$ with that of the square root, one can show that all the roots (or their real parts) of the biquadratic (A 2) will be negative if $b_0 > 0$, $\mathrm{Re}\,[\frac{1}{4}b_1 \pm M] > 0$ and $(\frac{1}{6}b_2 + 2b_0\theta \pm N)$ are real and positive, i.e. if

$$b_0 > 0, \quad \mathrm{Re}\,[(\tfrac{1}{4}b_1)^2 - M^2] > 0, \quad (\tfrac{1}{6}b_2 + 2b_0\theta)^2 - N^2 > 0,$$

i.e. if
$$b_0 > 0, \quad (\tfrac{1}{4}b_1)^2 > \mathrm{Re}\,(M^2), \quad (\tfrac{1}{6}b_2 + 2b_0\theta)^2 > N^2. \quad \text{(A 4)}$$

Since complex roots of a real polynomial always occur in conjugate pairs, the cubic equation (A 3) has at least one real solution, and from (A 2) M^2 is real. The sufficient condition described in (A 4) becomes

$$b_0 > 0, \quad \tfrac{1}{6}b_2 > b_0\theta, \quad b_0 b_4 > 0. \quad \text{(A 4)}'$$

From a change of variable $\xi = 6b_0\theta$, one can rewrite the cubic equation (A 3) as

$$\xi^3 + 3H\xi + G = 0, \quad \text{(A 5)}$$

where
$$H = -\tfrac{1}{4}(12b_0 b_4 - 3b_1 b_3 + b_2^2),$$
$$G = \tfrac{1}{4}b_2(36b_0 b_4 - b_2^2) + \tfrac{9}{8}(b_1 b_2 b_3 - 3b_0 b_3^2 - 3b_1^2 b_4).$$

Since from (A 4)$'$ $b_2 > \xi$, combining this inequality with (A 5) leads to

$$b_2^3 + 3Hb_2 + G > 0. \quad \text{(A 6)}$$

Substitution for H and G defined in (A 5) and using (A 4)$'$ and (A 6) lead to the desired sufficient condition; i.e., if

$$b_0 > 0, \quad b_1 b_2 b_3 - b_0 b_3^2 - b_1^2 b_4 > 0 \quad \text{and} \quad b_4 > 0,$$

all the roots (or their real parts) of the quartic (A 1) will be negative.

Appendix C. Necessary and sufficient condition that all the roots (or their real parts if complex) of a real cubic are negative

It is well known that if all the roots (or their real parts) of a real cubic,

$$x^3 + p_1 x^2 + p_2 x + p_3, \quad \text{(A 7)}$$

are negative, all its coefficients are positive (Burnside & Panton 1892). That the converse is also true will be shown by reductio ad absurdum. Since complex roots of a real polynomial occur in conjugate pairs, the cubic (A 7) has either one or three real roots. First consider that all the roots x_1, x_2 and x_3 of (A 7) are real. Suppose that $p_1 > 0$, $p_2 > 0$ and $p_3 > 0$.

Case (i): Clearly x_1, x_2 and x_3 cannot be all positive for this would imply $p_1 < 0$ and $p_3 < 0$.

Case (ii): If $x_1 = -\tau < 0$, $x_2 > 0$ and $x_3 > 0$, then, since $p_1 = -(-\tau + x_2 + x_3)$, $p_2 = -\tau(x_2+x_3)+x_2x_3$ and $p_3 = \tau x_2 x_3$, $p_1 > 0$ and $p_2 > 0$ imply that $\tau > x_2 + x_3$ and $x_2x_3 > \tau(x_2+x_3)$, hence $x_2x_3 > (x_2+x_3)^2$, i.e. $0 > x_2^2+x_3^2+x_2x_3$, which is a contradiction.

Case (iii): If $x_1 = -\tau < 0$, $x_2 = -\beta < 0$ and $x_3 > 0$, then, since $p_3 = -\tau\beta x_3 < 0$, this contradicts the hypothesis. Consequently the only remaining alternative will be that, if $p_1 > 0$, $p_2 > 0$, $p_3 > 0$, then $x_1 < 0$, $x_2 < 0$, $x_3 < 0$.

Next consider that x_1 is a real root and x_2 and x_3 are the complex conjugate roots. Suppose $p_1 > 0$, $p_2 > 0$ and $p_3 > 0$.

Case (i): If $x_1 > 0$, $\mathrm{Re}\,(x_2) = \mathrm{Re}\,(x_3) > 0$, then

$$p_1 = -[x_1 + 2\,\mathrm{Re}\,(x_2)] < 0 \quad \text{and} \quad p_3 = -x_1|x_2|^2 < 0.$$

This is a contradiction.

Case (ii): If $x_1 > 0$, $\mathrm{Re}\,(x_2) = \mathrm{Re}\,(x_3) < 0$, then

$$p_3 = -x_1|x_2|^2 < 0.$$

Again this is a contradiction.

Case (iii): Let $x_1 = -\tau < 0$, $x_2 = \xi + i\beta$, $x_3 = \xi - i\beta$, where ξ and β are real. There are now only two remaining alternatives: either $\xi < 0$ or $\xi > 0$. Since

$$p_1 = -(-\tau + 2\xi) = \tau - 2\xi, \tag{A 8}$$

$$p_2 = -\tau(2\xi) + \xi^2 + \beta^2, \tag{A 9}$$

$$p_3 = \tau(\xi^2 + \beta^2), \tag{A 10}$$

eliminating τ and β from (A 8), (A 9) and (A 10) leads to

$$p_3 = (p_1 + 2\xi)\,[p_2 + 2\xi(p_1 + 2\xi)].$$

Since $\xi \geqslant 0$ corresponds to $p_3 \geqslant p_1p_2$, one can show that $\xi < 0$ corresponds to $p_3 < p_1p_2$. Hence

$$p_1, p_2, p_3 > 0 \text{ and } p_1p_2 > p_3 \Leftrightarrow x_1 < 0, \mathrm{Re}\,(x_2) = \mathrm{Re}\,(x_3) < 0.$$

This result has been applied on pages 118 and 119 in the text for obtaining a criterion under which interfacial waves will be stable.

REFERENCES

BEURIGER, J. 1901 *Zur Auflösung der biquadratischen Gleichungen.* Program (Beilage) K Bonn: Gymnasium.

BURNSIDE, W. S. & PANTON, A. W. 1892 *Theory of Equations*, 3rd edn. Dublin University Press.

CHANDRASEKHAR, S. 1955 *Proc. Camb. Phil. Soc.* **51**, 162–178.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*, ch. 10. Oxford University Press.

HARRISON, W. J. 1908 *Proc. Lond. Math. Soc.* **6**, 396–405.
HSIEH, D. Y. 1972 *Trans. A.S.M.E.* D, *J. Basic Engng* **94**, 156–000.
HSIEH, D. Y. 1978 *Phys. Fluids* **21**, 745–748.
LEWIS, W. J. 1950 *Proc. Roy. Soc.* A **202**, 81–98.
PALMER, H. J. 1976 *J. Fluid Mech.* **75**, 487–511.
RAYLEIGH, LORD 1900 *Scientific Papers*, vol. 2, pp. 200–227. Cambridge University Press.
TAYLOR, G. I. 1950 *Proc. Roy. Soc.* A **201**, 192–196.